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INVESTIGATION OF THE SPECTRUM OF SHORT-WAVE GÖRTLER VORTICES IN A GAS

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The linear stage of short-wave Görtler vortices in the boundary layer near a concave surface is studied for the regime of weak hypersonic viscid-inviscid interaction at high Reynolds and Görtler numbers. It is assumed that the gas is perfect and the viscosity is a linear function of the enthalpy. It is found that neutral vortices are located near the surface if it has zero temperature. When the surface is heated, the vortices move away from it, whereas all newly incipient vortices are located near the surface. It is shown that the growth rate of the vortices has a maximum and the heating of the surface has a stabilizing effect on the vortices.

An asymptotic theory (for high Reynolds and Görtler numbers) of Görtler vortices [7] has been developed [1–6] for a liquid. The basic modes are studied in order of increasing wavelength of the vortices:

— neutral short-wave vortices that have risen into the main part of the boundary layer,

— near-wall short-wave vortices with a maximum growth rate,

— vortices with a wavelength comparable with the boundary-layer thickness,

— long-wave first mode, which induces a three-layer disturbed flow,

— long-wave neutral vortices with a maximum wavelength, for which the "growth" of the boundary layer should be taken into account.

Model boundary-value problems have been posed for all regimes, similarity parameters have been determined, and numerical or analytical solutions have been obtained in a linear approximation. Nonlinear solutions have been obtained for certain regimes [8–10].

The modern stage of development of hypersonic flying vehicles has initiated the study of Görtler vortices in a gas. These ordered vortex structures can significantly affect heat exchange in boundary layers and flow structures with a curvature (for example, due to flow reattachment [11]). In early papers, for instance [12], the effect of various flow parameters on the eigenvalues of linearized Navier–Stokes equations was studied. It has been found that the allowance for compressibility, an increase in viscosity, or an increase in the surface temperature have a stabilizing effect on the vortices, whereas adverse pressure gradients have an opposite effect. Obviously, for moderate free-stream Mach numbers, the structure of the vortices should not be significantly different from their structure in a liquid. Thus, Spall and Malik [13] and Wadey [14] studied long-wave vortices in a gas for which it is necessary to take into account the "growth" of the boundary layer. It has been found that unstable vortices shift toward the outer edge of the boundary layer as the Mach number increases [14]. Lipatov [15] and Bogolepov and Lipatov [16] studied an asymptotic structure of vortices with a wavelength comparable with or exceeding the boundary-layer thickness. It this case, the effects of varying gas density can be manifested. It is usually assumed, however, that the main difference of the hypersonic boundary layer from the boundary layer in a liquid is the presence of a temperature adjustment layer near the boundary-layer edge, where the temperature rapidly decreases from the deceleration value to its free-stream value [17–19]. Real gas properties were also taken into account in [19].

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Short-wave vortices in the near-wall portion of the boundary layer near a strongly cooled surface in the regime of weak hypersonic viscid-inviscid interaction [20] were studied in [21, 22]. It was shown that if the surface has zero temperature, the neutral vortices do not rise into the main portion of the boundary layer, and the normalized growth rate of the amplitude of the vortices has a maximum. When the surface temperature is different from zero, the neutral vortices move away from it.

1. The uniform viscous gas flow around a concave surface is considered for high but subcritical Reynolds numbers $\mathrm{Re}_\infty = \rho_\infty u_\infty L/\mu_\infty \gg 1$, i.e., it is assumed that the boundary-layer flow remains laminar. Here L is a certain streamwise distance measured from the leading edge of the surface; ρ_∞, u_∞, and μ_∞ are the free-stream density, velocity, and viscosity of the gas. It is supposed that the surface curvature is small $k = L/R \ll 1$ (R is the radius of the surface curvature). In what follows, only dimensionless variables are used. All linear dimensions are normalized to L, the pressure p and enthalpy h to $\rho_\infty u_\infty^2$ and u_∞^2, respectively, and the remaining stream functions to their free-stream values.

The free-stream Mach number is assumed to be rather high $\mathrm{M}_\infty \gg 1$. It is known [20] that the deceleration of a gas in a boundary layer at high supersonic free-stream velocities leads to very high temperatures in the boundary layer and a significant increase in its thickness. Thus, it is necessary to evaluate the pressure perturbation due to the displacing action of the boundary layer. It is assumed to be small as compared with the free-stream pressure $\Delta p_1 \sim \delta/\mathrm{M}_\infty \ll 1/\mathrm{M}_\infty^2$ (δ is the boundary-layer thickness). The pressure perturbation due to the surface curvature is also assumed to be small $\Delta p_2 \sim k/\mathrm{M}_\infty \ll 1/\mathrm{M}_\infty^2$. Hence, it follows that for stream functions in the boundary layer with characteristic dimensions $\Delta x \sim 1$ and $\Delta y \sim \delta$ (the x axis is directed streamwise along the surface and the y axis is normal to it) the following estimates for the regime of weak hypersonic viscid-inviscid interaction are valid [20]:

$$u \sim h \sim 1, \quad v \sim \delta, \quad p \sim \rho \sim \frac{1}{\mathrm{M}_\infty^2}, \quad \mu \sim \mathrm{M}_\infty^2, \quad \delta \sim \frac{\mathrm{M}_\infty^2}{\mathrm{Re}_\infty^{1/2}}, \quad C_f \sim C_q \sim \frac{\delta}{\mathrm{M}_\infty^2}. \tag{1.1}$$

Here u and v are the velocity components along the x and y axes, and C_f and C_q are the friction-stress and heat-flux coefficients. In obtaining estimates (1.1) we used the linear dependence of viscosity on enthalpy

$$\mu = A\mathrm{M}_\infty^2 h \tag{1.2}$$

and the equation of state for a perfect gas

$$\gamma p = (\gamma - 1)\rho h, \tag{1.3}$$

where A is a constant and γ is the ratio of specific heats.

Assuming the coefficients C_f and C_q to preserve the orders of magnitude in the near-wall portion of the boundary layer for $y/\delta \ll 1$, we obtain the following equation from (1.1) and (1.2):

$$\begin{gathered} C_q = \frac{\delta}{\mathrm{M}_\infty^2}\frac{B}{\mathrm{Pr}} = \frac{\mu}{\mathrm{Re}_\infty \mathrm{Pr}}\left(\frac{\partial h}{\partial y}\right)_w, \qquad C_f = \frac{\delta}{\mathrm{M}_\infty^2} C = \frac{\mu}{\mathrm{Re}_\infty}\left(\frac{\partial u}{\partial y}\right)_w, \\ h = \left(\frac{2B}{A}\frac{y}{\delta} + h_w^2\right)^{1/2}, \qquad u = \frac{C}{B}\left(\frac{2B}{A}\frac{y}{\delta} + h_w^2\right)^{1/2} - \frac{C}{B}h_w, \end{gathered} \tag{1.4}$$

where B and C are certain constants, h_w is the Prandtl number, and h_w is the enthalpy of the gas near the surface. For high supersonic velocities of the incoming flow, the value of h_w on the cooled surface is small ($h_w \ll 1$). This substantially alters the properties of the boundary layer compared with the case of finite values of h_w [23].

For finite values of h_w $[(y/\delta)^{1/2} \ll h_w \leqslant 1]$, from relations (1.4) we obtain

$$h \approx h_w + \frac{B}{Ah_w}\frac{y}{\delta}, \qquad u \approx \frac{C}{Ah_w}\frac{y}{\delta}. \tag{1.5}$$

Relations (1.5) show that in this case the near-wall portion of the boundary layer is isothermic and has a linear profile of the longitudinal velocity.

It is known that, under certain conditions, a two-dimensional laminar boundary layer near a concave surface can become unstable [7]. Then steady Görtler vortices extended in the streamwise direction are formed inside the boundary layer, and the two-dimensional flow becomes three-dimensional. This transition occurs when a certain critical value of the Görtler number $G_\infty = 2(\mathrm{Re}_\infty^{1/2}/M_\infty^2)(L/R)$ is exceeded. Below, we study a disturbed flow with high Görtler numbers $G_\infty \sim æ/\delta \gg 1$, $k = æK$, $K \sim 1$, and $æ \ll 1$, where the vortices certainly exist.

2. We study a disturbed vortex flow region with a characteristic thickness $\Delta y \ll \delta$ near the surface at a distance $\Delta x \sim 1$ from its leading edge, in which the vortices are localized. It is assumed that the loss of stability of the boundary layer caused by vortex formation induces nonlinear disturbances of stream functions in this region (for example, $\Delta u \sim u$). If we denote the orders of magnitude of the longitudinal velocity and enthalpy as u_0 amd h_0, then, taking into account (1.1), from Eqs. (1.2) and (1.3) we obtain

$$p \sim 1/M_\infty^2, \qquad \rho \sim 1/M_\infty^2 h_0, \qquad \mu \sim M_\infty^2 h_0. \tag{2.1}$$

A comparison of the orders of magnitude of the convective terms of the Navier–Stokes equations shows that the additional pressure perturbation

$$\Delta p \sim k\rho u^2 \Delta y \sim æ u_0^2 \Delta y / M_\infty^2 h_0 \tag{2.2}$$

arises in the field of centrifugal forces, and it induces a transverse velocity component w directed along the z axis perpendicular to the xy plane:

$$w \sim (\Delta p/\rho)^{1/2} \sim æ^{1/2} u_0 \Delta y^{1/2}. \tag{2.3}$$

Assuming that the transverse dimensions of the disturbed region along the y and z axes are generally equal in order of magnitude $\Delta y \sim \Delta z$, from the equation of continuity we obtain

$$v \sim w, \qquad \Delta y \sim \Delta z \sim æ\Delta x^2, \tag{2.4}$$

where Δx is the characteristic length of the disturbed region. A comparison of the orders of magnitude of the main convective and dissipative terms of the Navier–Stokes equations yields the expression

$$\Delta x^3 \sim \delta^2 h_0^2 / æ^2 u_0. \tag{2.5}$$

The estimates for the transverse velocity component w, which represents vortex disturbances of the boundary layer, are found from a comparison of the orders of magnitude of the convective terms of the Navier–Stokes equations. Thus, convection is the basic mechanism in the formation of Görtler vortices.

For small values of h_w [$0 \leqslant h_w \leqslant (\Delta y/\delta)^{1/2} \ll 1$], relations (1.4) and estimates (2.4) and (2.5) lead to the following estimates for u_0, h_0, and the characteristic dimensions of the disturbed region:

$$u_0 \sim h_0 \sim \left(\frac{\Delta y}{\delta}\right)^{1/2}, \qquad \Delta x \sim \left(\frac{\delta}{æ}\right)^{3/4} \ll 1, \qquad \Delta y \sim \Delta z \sim \delta\left(\frac{\delta}{æ}\right)^{1/2} \ll \delta. \tag{2.6}$$

If $(\Delta y/\delta)^{1/2} \ll h_w \leqslant 1$, then from (1.5), (2.4), and (2.5) it follows that

$$h_0 \sim h_w, \quad u_0 \sim \Delta y/\delta h_w, \quad \Delta x \sim (\delta h_w/æ)^{3/5} \ll 1, \quad \Delta y \sim \Delta z \sim \delta h_w (\delta h_w/æ)^{1/5} \ll \delta. \tag{2.7}$$

Estimates (2.6) and (2.7) show that surface heating leads to an increase in the dimensions of the vortices. For $h_w \sim 1$, from (2.7) we obtain estimates for the dimensions of the disturbed region in a liquid [3, 4, 6].

Estimates (2.1)–(2.5) allow the introduction of new variables and asymptotic expansions of the stream functions for the disturbed region:

$$\begin{gathered} x = (\delta^2 h_0^2/æ^2 u_0)^{1/3} x_1, \quad y = (\delta^4 h_0^4/æ u_0^2)^{1/3} y_1, \quad z = (\delta^4 h_0^4/æ u_0^2)^{1/3} z_1, \quad u = u_0 u_1 + \ldots, \\ v = (æ\delta^2 u_0^2 h_0^2)^{1/3} v_1 + \ldots, \quad w = (æ\delta^2 u_0^2 h_0^2)^{1/3} w_1 + \ldots, \quad \mu = M_\infty^2 h_0 \mu_1 + \ldots, \\ \rho = (1/M_\infty^2 h_0)\rho_1 + \ldots, \quad h = h_0 h_1 + \ldots, \quad p = 1/\gamma M_\infty^2 + (æ^{2/3}\delta^{4/3} u_0^{4/3} h_0^{1/3}/M_\infty^2) p_1 + \ldots . \end{gathered} \tag{2.8}$$

Substitution of (2.8) into the Navier–Stokes equations and into (1.2) and (1.3) and passage to the limit for $M_\infty \to \infty$, $\delta \to 0$, $\delta \ll æ \ll 1$, $M_\infty\delta \ll 1$, and $M_\infty æ \ll 1$ show that in the first approximation the disturbed flow region is described by the system of equations

$$(\rho_1 u_1)_{x_1} + (\rho_1 v_1)_{y_1} + (\rho_1 w_1)_{z_1} = 0, \quad \rho_1(u_1 u_{1x_1} + v_1 u_{1y_1} + w_1 u_{1z_1}) = (\mu_1 u_{1y_1})_{y_1} + (\mu_1 u_{1z_1})_{z_1},$$
$$\rho_1(u_1 v_{1x_1} + v_1 v_{1y_1} + w_1 v_{1z_1} + K u_1^2) + p_{1y_1} = (\mu_1 v_{1y_1})_{y_1} + (\mu_1 v_{1z_1})_{z_1},$$
$$\rho_1(u_1 w_{1x_1} + v_1 w_{1y_1} + w_1 w_{1z_1}) + p_{1z_1} = (\mu_1 w_{1y_1})_{y_1} + (\mu_1 w_{1z_1})_{z_1}, \tag{2.9}$$
$$\mathrm{Pr}\rho_1(u_1 h_{1x_1} + v_1 h_{1y_1} + w_1 h_{1z_1}) = (\mu_1 h_{1y_1})_{y_1} + (\mu_1 h_{1z_1})_{z_1}, \quad (\gamma - 1)\rho_1 h_1 = 1, \quad \mu_1 = A h_1.$$

On the surface we set the usual conditions for the velocity components and enthalpy:

$$u_1 = v_1 = w_1 = 0, \qquad h_1 = h_{1w} \qquad (y_1 = 0). \tag{2.10}$$

Here $h_{1w} = h_w/h_0$, and the external and initial boundary conditions are obtained from matching with the solution for the near-wall portion of the boundary layer (1.4)

$$u_1 \to \frac{C}{B}\left(\frac{2B}{A}y_1 + h_{1w}^2\right)^{1/2} - \frac{C}{B}h_{1w}, \ h_1 \to \left(\frac{2B}{A}y_1 + h_{1w}^2\right)^{1/2}, \ \rho_1 \to 1/(\gamma-1)\left(\frac{2B}{A}y_1 + h_{1w}^2\right)^{1/2},$$
$$\mu_1 \to A\left(\frac{2B}{A}y_1 + h_{1w}^2\right)^{1/2}, \quad v_1, w_1 \to 0, \quad p_{1y_1} \to -K\rho_1 u_1^2 \quad (x_1 \to -\infty \text{ or } y_1 \to \infty). \tag{2.11}$$

The periodicity condition is set in the transverse direction:

$$f(x_1, y_1, z_1) = f(x_1, y_1, z_1 + \lambda), \qquad f = u_1, v_1, w_1, p_1, \rho_1, h_1, \mu_1 \tag{2.12}$$

(λ is the wavelength of the vortices).

The solution of the boundary-value problem (2.9)–(2.12) describes short-wave Görtler vortices in the near-wall portion of the boundary layer in a liquid or in a gas with $\Delta y \sim \Delta z \ll \delta$ and $\Delta x \ll 1$ depending on the value of h_{1w}. The evolution of the vortices in the first approximation proceeds in a plane-parallel flow, since for a small distance ($\Delta x \ll 1$) the longitudinal variation in the stream functions in the boundary layer is insignificant.

In what follows, it is convenient to normalize the variables x_1, y_1, z_1, u_1, v_1, w_1, p_1, ρ_1, h_1, and μ_1 to the following quantities: $(\lambda/2\pi K)^{1/2}$, $\lambda/2\pi$, $\lambda/2\pi$, $C(\lambda/AB\pi)^{1/2}$, $(K/2AB)^{1/2}C\lambda/\pi$, $(K/2AB)^{1/2}C\lambda/\pi$, $(\lambda^3/AB^3\pi^3)^{1/2}KC^2/2(\gamma-1)$, $(A\pi/B\lambda)^{1/2}/(\gamma-1)$, $(B\lambda/A\pi)^{1/2}$, and $(AB\lambda/\pi)^{1/2}$. In the new variables (without the subscript 1) the boundary-value problem (2.9)–(2.12) is transformed into

$$(\rho u)_x + (\rho v)_y + (\rho w)_z = 0, \quad \mathrm{Re}\rho(uu_x + vu_y + wu_z) = (\mu u_y)_y + (\mu u_z)_z,$$
$$\mathrm{Re}[\rho(uv_x + vv_y + wv_z + u^2) + p_y] = (\mu v_y)_y + (\mu v_z)_z,$$
$$\mathrm{Re}[\rho(uw_x + vw_y + ww_z) + p_z] = (\mu w_y)_y + (\mu w_z)_z,$$
$$\mathrm{Re}\,\mathrm{Pr}\rho(uh_x + vh_y + wh_z) = (\mu h_y)_y + (\mu h_z)_z, \quad \rho h = 1, \quad \mu = h, \tag{2.13}$$
$$u = v = w = 0, \qquad h = D \quad (y = 0),$$
$$u \to h_0 - D, \qquad v, w \to 0, \qquad \rho \to 1/h_0, \qquad \mu, h \to h_0,$$
$$p \to -\frac{2}{3}(h_0^3 - D^3) + 2Dy - 2D^2(h_0 - D) \qquad (x \to -\infty \text{ or } y \to \infty),$$
$$f(x, y, z) = f(x, y, z + 2\pi), \qquad f = u, v, w, p, \rho, h, \mu,$$
$$\mathrm{Re} = \frac{CK^{1/2}\lambda}{2^{3/2}\pi A^{1/2}B^{3/2}(\gamma-1)}, \quad D = h_{1w}\left(\frac{A\pi}{B\lambda}\right)^{1/2}, \quad h_0(y) = (y + D^2)^{1/2},$$

where Re is the local Reynolds number, D is the reduced enthalpy of the gas near the surface, and $h_0(y)$ is the enthalpy profile in the near-wall portion of the undisturbed boundary layer.

For high values of the parameter $D \gg y_*^{1/2}$ (y_* is a certain thickness of the disturbed region), the external and initial boundary conditions (2.13) are transformed into

$$u \to \frac{y}{2D}, \quad v, w \to 0, \quad \rho \to \frac{1}{D} - \frac{y}{2D^3}, \quad \mu, h \to D + \frac{y}{2D}, \quad p \to -\frac{y^3}{12D^3}. \tag{2.14}$$

It is seen from these expressions [see also relations (1.5)] that the flow is incompressible and isothermic with constant viscosity. It can be easily seen that, if we renormalize the variables, the boundary-value problem (2.13) is transformed to formulas that model short-wave vortices in the near-wall portion of the boundary layer in a liquid [3, 4, 6], in which the only similarity parameter is the local Reynolds number for the liquid Re_*:

$$\mathrm{Re}_* = \mathrm{Re}/2D^3 \sim K^{1/2}(\lambda/2\pi)^{5/2}. \tag{2.15}$$

3. For small perturbations of the original boundary-layer flow, formulas (2.13) can be linearized with respect to the initial boundary conditions

$$u = h_0 - D + \alpha U + \ldots, \quad v = \alpha V + \ldots, \quad w = \alpha W + \ldots, \quad h = h_0 + \alpha H + \ldots, \quad \mu = h_0 + \alpha H + \ldots,$$
$$p = -\frac{2}{3}(h_0^3 - D^3) + 2Dy - 2D^2(h_0 - D) + \alpha P + \ldots, \quad \rho = \frac{1}{h_0} - \alpha\frac{H}{h_0^2} + \ldots, \tag{3.1}$$

where $\alpha \ll 1$ and the final relations for ρ and μ from (2.13) are already taken into account. In the new variables (3.1) the boundary-value problem (2.13) takes the form

$$2h_0^2(U_x + V_y + W_z - H_x) + 2Dh_0H_x - V = 0,$$
$$2\mathrm{Re}\, h_0[2h_0(h_0 - D)U_x + V] = 4h_0^4(U_{yy} + U_{zz}) + 2h_0^2(U_y + H_y) - H,$$
$$2\mathrm{Re}\, h_0\left[\frac{h_0 - D}{h_0}V_x + 2\frac{h_0 - D}{h_0}U - \frac{(h_0 - D)^2}{h_0^2}H + P_y\right] = 2h_0^2(V_{yy} + V_{zz}) + V_y,$$
$$2\mathrm{Re}\, h_0\left[\frac{h_0 - D}{h_0}W_x + P_z\right] = 2h_0^2(W_{yy} + W_{zz}) + W_y, \tag{3.2}$$
$$2\mathrm{Re}\,\mathrm{Pr} h_0[2h_0(h_0 - D)H_x + V] = 4h_0^4(H_{yy} + H_{zz}) + 4h_0^2H_y - H,$$
$$U, V, W, H, P \to 0 \qquad (y = 0 \text{ or } x \to -\infty \text{ or } y \to \infty),$$
$$F(x, y, z) = F(x, y, z + 2\pi), \qquad F = U, V, W, H, P.$$

The boundary-value problem (3.2) allows a normal-mode representation of the solution [24]:

$$F(x, y, z) = F_1(y)\exp(\beta x)(\sin z, \cos z).$$

This allows us to transform the equations in partial derivatives (3.2) into ordinary differential equations

$$2h_0^2(\beta U_1 + V_1' + W_1 - \beta H_1) + 2D\beta h_0 H_1 - V_1 = 0,$$
$$2\mathrm{Re}\, h_0[2h_0(h_0 - D)\beta U_1 + V_1] = 4h_0^4(U_1'' - U_1) + 2h_0^2(U_1' + H_1') - H_1,$$
$$2\mathrm{Re}\, h_0\left[\frac{h_0 - D}{h_0}\beta V_1 + 2\frac{h_0 - D}{h_0}U_1 - \frac{(h_0 - D)^2}{h_0^2}H_1 + P_1'\right] = 2h_0^2(V_1'' - V_1) + V_1',$$
$$2\mathrm{Re}\, h_0\left[\frac{h_0 - D}{h_0}\beta W_1 - P_1\right] = 2h_0^2(W_1'' - W_1) + W_1', \tag{3.3}$$
$$2\mathrm{Re}\,\mathrm{Pr}\, h_0[2h_0(h_0 - D)\beta H_1 + V_1] = 4h_0^4(H_1'' - H_1) + 4h_0^2H_1' - H_1,$$
$$U_1(0) = V_1(0) = W_1(0) = H_1(0) = U_1(\infty) = V_1(\infty) = W_1(\infty) = H_1(\infty) = P_1(\infty) = 0.$$

The solution of the boundary-value problem (3.3) is its eigenfunctions that correspond to the values of the parameters Re and β (for instance, for fixed values of the parameters Pr and D).

4. For high values of the local Reynolds number $\mathrm{Re} \gg 1$, the dissipative terms in (3.3) become insignificant, and only nonpenetration conditions can be satisfied on the surface. In this case, the boundary-

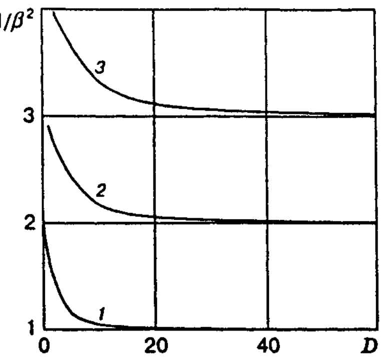

Fig. 1

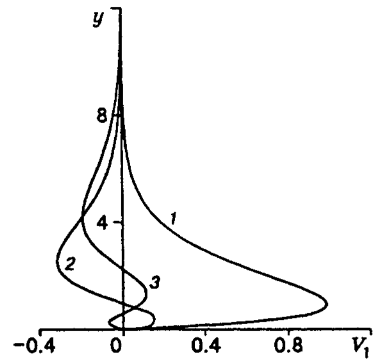

Fig. 2

value problem (3.3) is substantially simplified, and it can be reduced to the following equation for the function $V_1(y)$:

$$V_1'' - \frac{V_1'}{2h_0^2} + \left[\frac{h_0 + D}{2h_0^2(h_0 - D)\beta^2} + \frac{1}{2h_0^3(h_0 - D)} - 1\right] V_1 = 0, \quad V_1(0) = V_1(\infty) = 0. \tag{4.1}$$

For high values of the parameter $D \gg y_*^{1/2}$, this equation is transformed to the equation for a liquid [3, 4, 6], for which the following analytical dependence is obtained:

$$1/\beta^2 = n, \qquad n = 1, 2, 3\ldots \tag{4.2}$$

(n is the mode number). The eigenvalues of (4.1) were calculated by the method of inverse iterations with a shift [25]. Figure 1 shows the values of $1/\beta^2$ versus D for the first three modes (curves 1–3). It is seen that even for $D = 60$ the calculation results differ insignificantly from exact analytical solutions (4.2). For these values of D and $\mathrm{Re} \gg 1$, the gas can be already considered incompressible and isothermic. To estimate the growth rate of the amplitude of the vortices, it is advisable to use the growth rate normalized to the characteristic length $\Delta x \sim 1$:

$$\mathrm{Be} = (1/\Delta x)\beta(2\pi K/\lambda)^{1/2}. \tag{4.3}$$

It is seen from Fig. 1 that the value of β for all modes increases slightly with increase in D. It follows from estimates (2.7), however, that Δx and λ increase by an order of magnitude, i.e., as D increases when the surface is heated, the value of Be decreases for $\mathrm{Re} \gg 1$. The surface heating has a stabilizing effect on the vortices, which is explained by an increase in the dimensions of the disturbed region and a decrease in vorticity of the incoming flow [see relations (2.7) and (2.14)].

It was found in calculations that for a zero temperature of the surface $D = 0$, the eigenvalue of the boundary-value problem (4.1) exists only for the first mode. As D increases to $D \approx 1$, it is possible to find a second mode. And further, as D increases, higher modes are obtained. This can be interpreted as excitation of higher degrees of freedom of a certain system as its temperature increases. For $D = 0$, Eq. (4.1) admits the analytical solution

$$V_1(1/\beta^2, y) = y\exp(-y)\left[{}_1F_1\left(\frac{3 - 1/\beta^2}{4}, \frac{3}{2}, 2y\right) - \left(\Gamma\left(\frac{1 - 1/\beta^2}{4}\right)\Big/ 2\Gamma\left(\frac{3 - 1/\beta^2}{4}\right)\sqrt{2y}\right){}_1F_1\left(\frac{1 - 1/\beta^2}{4}, \frac{1}{2}, 2y\right)\right],$$

where ${}_1F_1(a, b, x)$ is the Pochhammer degenerate hypergeometrical function and $\Gamma(x)$ is the gamma function. It can easily be shown that this solution has only one extremum, i.e., it represents only the first mode. Figure 2 shows calculated profiles of the function $V_1(y)$ for the first three modes for $D = 5$.

5. For numerical integration of the boundary-value problem (3.3), we use new dependent variables $U_* = U_1$, $V_* = \mathrm{Re}V_1$, $W_* = \mathrm{Re}W_1$, $P_* = \mathrm{Re}^2 P_1$, and $H_* = H_1$. The functions W_* and P_* are excluded from the number of variables, and the boundary-value problem (3.3) is transformed to

$$4h_0^4 U_*'' + 2h_0^2 U_*' - 4h_0^2[h_0^2 + \beta_*(h_0 - D)]U_* = 2h_0 V_* - 2h_0^2 H_*' + H_*,$$

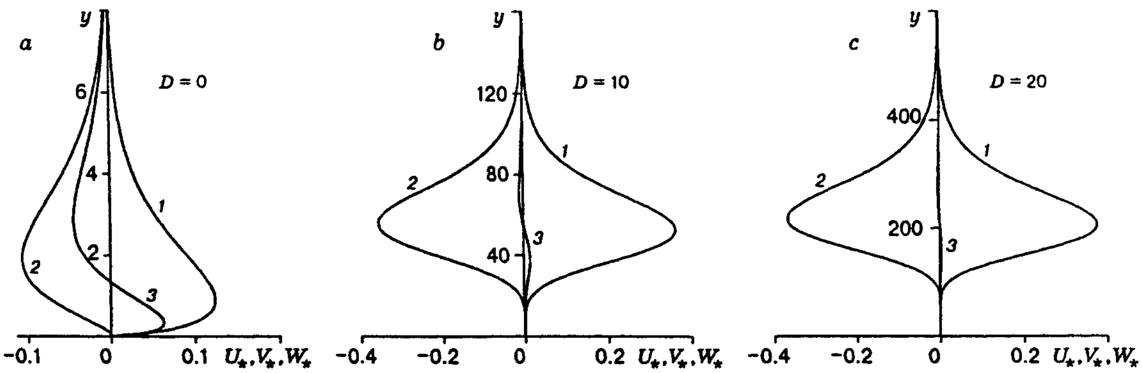

Fig. 3

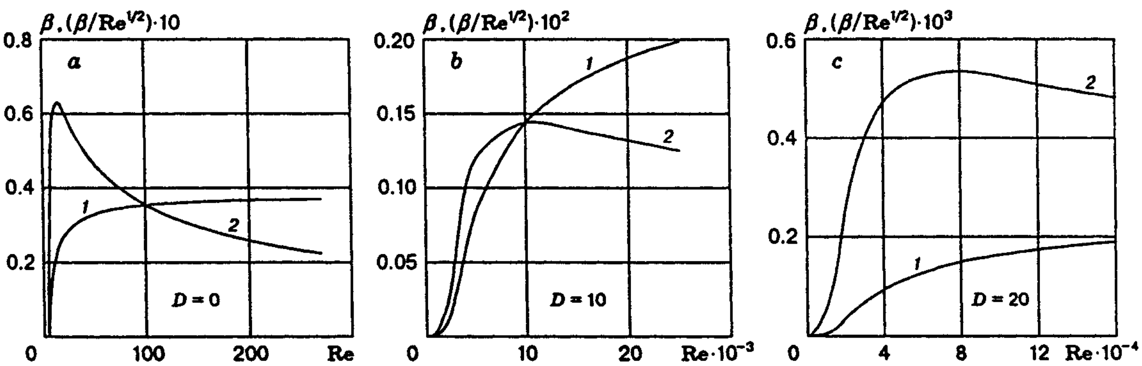

Fig. 4

$$4h_0^4 H_*'' + 4h_0^2 H_*' - [4h_0^4 + 1 + 4\mathrm{Pr}\beta_* h_0^2(h_0 - D)]H_* = 2\mathrm{Pr} h_0 V_*,$$

$$-4h_0^4 V_*'''' - 2h_0^2 V_*''' + [8h_0^4 + 4\beta_* h_0^2(h_0 - D) - 3]V_*'' + \left[2h_0^2 - 2\beta_*(2 - \mathrm{Pr})(h_0 - D) + \frac{15}{2h_0^2}\right]V_*'$$

$$+ \left[1 - \frac{15}{2h_0^4} - 4h_0^4 - 4\beta_* h_0^2(h_0 - D) + \frac{4\beta_*}{h_0}\left(1 - \frac{\mathrm{Pr}}{2}\right) + \frac{3\beta_* D}{h_0^2}(2\mathrm{Pr} - 1)\right]V_* \qquad (5.1)$$

$$= 4\mathrm{G}h_0^2(h_0 - D)\left(U_* - \frac{h_0 - D}{2h_0}H_*\right) + \left[\frac{15D\beta_*}{h_0} + 4\beta_*^2(1 - \mathrm{Pr})h_0(h_0 - D)^2\right]H_*'$$

$$- \beta_* D\left[6h_0 + \frac{15}{2h_0^3} + 2\beta_*\frac{h_0 - D}{h_0}(5\mathrm{Pr} - 2)\right]H_*,$$

$$U_*(0) = V_*(0) = V_*'(0) = U_*(\infty) = V_*(\infty) = V_*'(\infty) = 0$$

($\beta_* = \mathrm{Re}\,\beta$ and the local Görtler number is $\mathrm{G} = 2\mathrm{Re}^2$). To determine the eigenvalues of the boundary-value problem (5.1), we used the same method of inverse iterations [25]. From the known functions U_*, V_*, and H_* we determined the function W_* from the equation

$$W_* = \frac{V_*}{2h_0^2} - V_*' - \beta_*\left(U_* - \frac{h_0 - D}{h_0}H_*\right).$$

In the calculations we prescribed values of the Prandtl number Pr and the parameter D, and solving (5.1), determined the eigenfunctions U_*, V_*, and H_*, and the eigenvalue of G for various values of β_*. All calculations were performed only for the first mode. To calculate higher modes, it is necessary to reduce system (5.1) to one equation and use the previous method. This requires great analytical and calculation effort, and the results obtained will not be of principal importance.

Figure 3 shows profiles of the functions U_*, V_*, and W_* (curves 1–3) for neutral vortices ($\beta_* = 0$) for $D = 0$, 10, and 20 and Pr = 1 (it can easily be seen that in this case $U_* = H_*$). For $D = 0$ (Fig. 3a) the

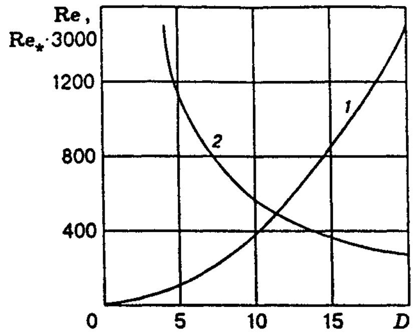

Fig. 5

vortices are located directly on the surface and do not rise into the main portion of the boundary layer, as occurs in a liquid. As D increases, the neutral vortices move away from the surface, their vertical dimensions increase (the vortices become flattened in the transverse direction), and the flow near the surface remains undisturbed (Fig. 3b and c). The calculations show that as β_* increases, the vortices that have risen gradually approach the surface.

Figure 4 shows the growth rate β (curves 1) and the quantity $(\beta/\mathrm{Re}^{1/2}) \cdot 10^m$, proportional to the reduced growth rate Be (curves 2),

$$\mathrm{Be} = \frac{1}{\Delta x}\beta\Big(\frac{2\pi K}{\lambda}\Big)^{1/2} \sim \frac{\beta}{\mathrm{Re}^{1/2}},$$

as functions of the local Reynolds number Re for $D = 0$, 10, and 20 ($m = 1$, 2, and 3, respectively) and $\mathrm{Pr} = 1$. The value of β increases monotonically, and for $\mathrm{Re} \gg 1$ it should approach its asymptotic value determined by solution of boundary-value problem (4.1) (see Fig. 1). The quantity $\beta/\mathrm{Re}^{1/2}$ should obviously have a maximum, since $\beta = 0$ on one end of the curves and $\mathrm{Re} \to \infty$ as $\beta \to$ const on the other. The bending of the curves in Fig. 4b and c near the coordinate origin can be attributed to flow rearrangement: the vortices go down here and approach the surface. From the picture of distributions of $\beta/\mathrm{Re}^{1/2}$, it can be concluded that as D increases (the surface is heated), the maximum point shifts to the region of high values of the local Reynolds numbers, and the value of the extremum itself decreases significantly. This proves a stabilizing effect of surface heating on the vortices for finite values of Re as well (see Sec. 4).

Figure 5 shows values of Re and $\mathrm{Re}_* \cdot 3000$ for neutral vortices [see (2.15)] versus D (curves 1 and 2). It is seen that surface heating leads to a dramatic increase in the minimum value of the local Reynolds number that makes the flow unstable, i.e., it has a strong stabilizing effect on the process of incipience of the vortices. Apart from the increase in vortex dimensions and decrease in vorticity in the incoming flow, surface heating is also manifested in the rising of vortices into the main portion of the boundary layer, in which the vorticity is even smaller than it is near the surface. A change in the Prandtl number Pr from 0.7 to 1.0 leads to an approximately 1.5% increase in the values of Re, which is not visible in Fig. 5. A drastic decrease of Re_* shows that, as D increases, the flow approaches a state typical of a liquid ($\mathrm{Re}_* \to 0$ as $\beta \to 0$ [3, 4, 6]).

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